QUADRATIC FORMS AND APPLICATIONS
TO GEOMETRY

by

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INTRODUCTION

One of the basic problems of plane analytic geometry is reducing the general equation of a second degree curve to canonical form by transforming to a new coordinate system. An analogous problem can be stated for a space of any dimensions. The solution of this and related problems is one of the fundamental aims of the theory of quadratic forms.

Rather than proceeding directly to quadratic forms, a method for changing matrices to diagonal form will be presented. With these results, a procedure for changing quadratic forms to canonical form is developed. Then utilizing the quadratic forms and the methods initiated, information sufficient for the complete identification of a second degree curve is derived. Finally, the extension of these methods to spaces of higher dimensions is indicated.

In this report, matrices are denoted by capital letters and their elements by lower case letters. If A is a matrix, the determinant of A is denoted by $|A|$, the transpose by A', and the conjugate by $\bar{A}$. The inverse of A, when it exists, is denoted by A^{-1}.

DEFINITIONS

The following definitions will be essential throughout this report.

Definition 1. A square matrix is said to be diagonal if the only nonzero elements are on the diagonal.

Definition 2. The characteristic polynomial of a matrix A is $f(\lambda) = |A - \lambda I|$. The characteristic equation of A is $|A - \lambda I| = 0$, and the zeros of the characteristic polynomial are the characteristic roots of A.

Definition 3. If A and B are two matrices for which there exists a nonsingular matrix P such that $P^{-1}AP = B$, then A is said to be similar to B.

Although the following definitions could be presented for vectors over either the complex or real field, only those defined over the real number field will be discussed in this report.

Definition 4. A quadratic homogeneous form is a function $f(X) = X'AX$, where $X \in V \leq V_n(F)$, A is a symmetrix matrix, X is a column vector, and V is a subspace of the vector space $V_n(F)$ of dimension n over the field F.

A matrix can be thought of as either an ordered set of row vectors or an ordered set of column vectors. It can be shown that each of these sets will span a vector space.

Definition 5. The row space of a matrix A is the vector space spanned by its row vectors. The column space of a matrix A is the vector space spanned by its column vectors.

Definition 6. A set of nonzero vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ is said to be a basis of a vector space V is:

1. The $\alpha_i \in V$ for $i = 1, 2, \ldots, n$.
2. Any vector in V can be expressed as a linear combination of the α_i.
3. The set is linearly independent.

Definition 7. A mapping f of VXV onto F (the real field), is called an inner product if, for all vectors $\alpha, \delta, \in V$, and $k \in F$,

1. $f(\alpha, \alpha)$ is nonnegative, $f(\alpha, \alpha) = 0$ implies $\alpha = 0$
2. $f(\alpha, \delta) = f(\delta, \alpha)$
3. $f(\alpha, k_1\delta_1 + k_2\delta_2) = k_1 f(\alpha, \delta_1) + k_2 f(\alpha_1, \delta_2)$.

If a vector space V over F has an inner product function defined, V is called a Euclidean space.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a basis for the vector space V such that $f(\alpha_i, \alpha_j) = \delta_{ij}$ (where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise), then V is said to be orthonormal.

Definition 8. A matrix S formed from an orthonormal basis of $V_n(F)$ has the property $SS' = S'S = I$. This matrix is called an orthogonal matrix.

TRANSFORMING A MATRIX TO DIAGONAL FORM

There are numerous techniques which can be used to change a matrix to diagonal form, some of which are more practical and efficient than the procedure to be used in the following. The method chosen, however, is convenient in that the only matrix theory required is relevant to the subsequent discussions.

By imposing various restrictions on a given matrix A, it can be shown that A will be similar to a diagonal matrix. Let the diagonal matrix be D. Then if A is similar to D, there exists a nonsingular matrix P such that $P^{-1}AP = D$. This can also be written as $AP \doteq PD$.

Using the equation AP = PD, a general method for finding the matrix P can be established. For the sake of computational simplicity the development is restricted to the case where A, P, and D are three-by-three matrices. The extension to matrices of higher order is readily apparent.

In matrix form AP = PD is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} .$$

Upon carrying out the matrix multiplication,

$$\begin{bmatrix} a_{11}p_{11} + a_{12}p_{21} + a_{13}p_{31} & a_{11}p_{12} + a_{12}p_{22} + a_{13}p_{32} & a_{11}p_{13} + a_{12}p_{23} + a_{13}p_{33} \\ a_{21}p_{11} + a_{22}p_{21} + a_{23}p_{31} & a_{21}p_{12} + a_{22}p_{22} + a_{23}p_{32} & a_{21}p_{13} + a_{22}p_{23} + a_{23}p_{33} \\ a_{31}p_{11} + a_{32}p_{21} + a_{33}p_{31} & a_{31}p_{12} + a_{32}p_{22} + a_{33}p_{32} & a_{31}p_{13} + a_{32}p_{23} + a_{33}p_{33} \end{bmatrix}$$

$$= \begin{bmatrix} p_{11}\lambda_1 & p_{12}\lambda_2 & p_{13}\lambda_3 \\ p_{21}\lambda_1 & p_{22}\lambda_2 & p_{23}\lambda_3 \\ p_{31}\lambda_1 & p_{32}\lambda_2 & p_{33}\lambda_3 \end{bmatrix} .$$

Equating corresponding elements of the matrix equation gives the following three systems of linear homogeneous equations in the p_{ij}.

$$\left.\begin{aligned} (a_{11} - \lambda_1)p_{11} + a_{12}p_{21} + a_{13}p_{31} &= 0 \\ a_{21}p_{11} + (a_{22} - \lambda_1)p_{21} + a_{23}p_{31} &= 0 \\ a_{31}p_{11} + a_{32}p_{21} + (a_{33} - \lambda_1)p_{31} &= 0 \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} (a_{11} - \lambda_2)p_{12} + a_{12}p_{22} + a_{13}p_{32} &= 0 \\ a_{21}p_{12} + (a_{22} - \lambda_2)p_{22} + a_{23}p_{32} &= 0 \\ a_{31}p_{12} + a_{32}p_{22} + (a_{33} - \lambda_2)p_{32} &= 0 \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} (a_{11} - \lambda_3)p_{13} + a_{12}p_{23} + a_{13}p_{33} &= 0 \\ a_{21}p_{13} + (a_{22} - \lambda_3)p_{23} + a_{23}p_{33} &= 0 \\ a_{31}p_{13} + a_{32}p_{23} + (a_{33} - \lambda_3)p_{33} &= 0 \end{aligned}\right\} \quad (3)$$

It should be noted that the coefficient matrices of these three systems differ only in the subscript of λ. Since these are homogeneous equations, nonzero solutions exist for these

systems if and only if

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

for $\lambda = \lambda_i$, $i = 1, 2, 3$. This is the characteristic equation of A and λ_1, λ_2, and λ_3 are the characteristic roots of A.

Using these roots, the systems of equations (1), (2), and (3) can be solved to find the column vectors of P. Note that this solution is not unique. If P is nonsingular, the equation $AP = PD$ can be written as $P^{-1}AP = D$. Therefore A is similar to a diagonal matrix.

QUADRATIC FORMS AND THEIR SIMPLIFICATION

Utilizing the concepts of orthogonality and similarity, a procedure can be derived for reducing a matrix to diagonal form.

Consider a homogeneous quadratic form in three variables x, y, and z. This is an expression of the form.

$$Q = ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy \ .$$

Writing this in matrix form, one has

$$Q = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or, denoting the coefficient matrix by A,

$$Q = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix} .$$

This equation can be written as

$$Q = \begin{bmatrix} x & y & z \end{bmatrix} IAI \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where I is the identity matrix. If S is an orthogonal matrix, then $SS' = I$ so that

$$Q = \begin{bmatrix} x & y & z \end{bmatrix} SS'ASS' \begin{bmatrix} x \\ y \\ z \end{bmatrix} .$$

Since $(XS)' = S'X'$,

$$(\begin{bmatrix} x & y & z \end{bmatrix} S)' = S' \begin{bmatrix} x \\ y \\ z \end{bmatrix} .$$

Let

$$\begin{bmatrix} x & y & z \end{bmatrix} S = \begin{bmatrix} X & Y & Z \end{bmatrix} ,$$

then

$$Q = \begin{bmatrix} X & Y & Z \end{bmatrix} S'AS \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} .$$

Before proceeding, the following theorem is needed.

Theorem 1. Every symmetric matrix A is orthogonally similar to a diagonal matrix (2).

The theorem implies that there exists a nonsingular matrix P such that $P^{-1}AP = D$, where A is symmetric and P is orthogonal so that $P' = P^{-1}$.

From this theorem it is seen that the orthogonal matrix S above can be chosen such that $S'AS = D$, where D is a diagonal matrix. In this case the quadratic form written in terms of X, Y, and Z, is said to be the canonical form. Then, in explicit form,

$$Q = \lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 \ .$$

This canonical form is not uniquely determined, as the ordering of the diagonal elements of D can be changed. In this report the elements will be ordered with respect to their absolute values, with those least in magnitude occurring first.

As an illustration of the above procedure, consider the quadratic equation

$$2x^2 + 4xy + 5y^2 = 1 \ .$$

It is desired to rotate the coordinate system so that the equation of the conic has no xy term. In matrix notation the equation becomes,

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \ .$$

To find the canonical form, the characteristic equation of the coefficient matrix must be found. The characteristic matrix is

$$\begin{bmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix}$$

and the characteristic equation is

$$\begin{vmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0 ,$$

or

$$\lambda^2 - 7\lambda + 6 = 0 .$$

The solutions of this equation are $\lambda = 6$ and $\lambda = 1$, so the canonical form for the equation is

$$X^2 + 6Y^2 = 1 .$$

The matrix of this rotation can be found by referring to the systems of equations (1), (2), and (3). The systems of equations become

$$\begin{cases} (2-\lambda_1)p_{11} + 2p_{21} = 0 \\ 2p_{11} + (5-\lambda_1)p_{21} = 0 \end{cases}$$

and

$$\begin{aligned} (2-\lambda_2)p_{12} + 2p_{22} &= 0 \\ 2p_{12} + (5-\lambda_2)p_{22} &= 0 \end{aligned} .$$

Substituting in the values $\lambda_1 = 1$ and $\lambda_2 = 6$, the systems become

$$\begin{cases} p_{11} + 2p_{21} = 0 \\ 2p_{11} + 4p_{21} = 0 \end{cases}$$

and

$$\begin{cases} -4p_{12} + p_{22} = 0 \\ 2p_{12} - p_{22} = 0 \end{cases} .$$

The solutions for these systems can be expressed as $p_{11} = -2p_{21}$ and $p_{22} = 2p_{12}$.

Although there are many choices available for the p's, they must be chosen so that the matrix P will be orthogonal; that is, $P'P = I$.

Since the vectors of P must be normal, it must be true that $p_{11}^2 + p_{21}^2 = 1$, so $4p_{21}^2 + p_{21}^2 = 1$ or $p_{21} = \frac{1}{\sqrt{5}}$, so $p_{11} = -\frac{2\sqrt{5}}{5}$. Similarly, $p_{12}^2 + p_{22}^2 = 1$, implies $p_{12}^2 + 4p_{12}^2 = 1$ or $p_{12} = \frac{1}{\sqrt{5}}$ so $p_{22} = \frac{2\sqrt{5}}{5}$. Thus the orthogonal matrix P is

$$\begin{bmatrix} -\frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{bmatrix}$$

It is then apparent that the choice of values to make P orthogonal should be $p_{21} = \sqrt{5}/5$ and $p_{12} = \sqrt{5}/5$.

To verify that the matrix P gives the desired rotation, compute the product

$$\begin{bmatrix} -\frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -\frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} .$$

Hence in this case the canonical form is

$$Q = \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = X^2 + 6Y^2 .$$

The reduction of a quadratic form to canonical form can be generalized to apply to conics whose representations involve n-dimensional vectors and matrices. In actual practice, due to the computation involved, finding the orthogonal matrix can be a formidable task.

HOMOGENEOUS COORDINATES AND CHANGE OF COORDINATE SYSTEMS

For the reduction of a quadratic form, it will be convenient to designate a vector X in E^n by an ordered set of n + 1 scalars $[X_1, X_2, \ldots, X_{n+1}]$ with $X_{n+1} \neq 0$, where the set denotes the vector

$$X = \left[\frac{X_1}{X_{n+1}}, \frac{X_2}{X_{n+1}}, \ldots, \frac{X_n}{X_{n+1}}\right]$$

The set of scalars $[X_1, X_2, \ldots, X_{n+1}]$ is called a set of <u>homogeneous</u> <u>coordinates</u> of X.

If homogeneous coordinates are introduced into polynomial equations in two or three variables, the equations become homogeneous in three or four variables, respectively. To illustrate the use of homogeneous coordinates, consider the second degree equation in two variables $ax^2 + by + c = 0$. Homogeneous coordinates are introduced by replacing x by x/w and y by y/w. The new equation is multiplied by w^2 to obtain $ax^2 + byw + cw^2 = 0$. This equation is homogeneous in the set of variables x, y, and w.

A transformation of coordinates in E^2 is a combination of a translation and a rotation about the origin. Suppose there exist two rectangular coordinate systems in the plane, with origins at O and O' and with the first and second coordinate axes parallel in the two systems. Then the point P in the plane has two sets of coordinates (x, y) and (x', y') with respect to the two sets of coordinate axes. If the coordinates of O' are (x_1, y_1) with respect to the old axes, then the coordinates (x, y) and (x', y') are related by the equations

$$\begin{cases} x = x' + x_1 \\ y = y' + y_1 \end{cases} .$$

These are called the translation equations.

The equations for rotating the axes about the origin through an angle θ are

$$\begin{cases} x' = x'' \cos\theta - y'' \sin\theta \\ y' = x'' \sin\theta + y'' \cos\theta \;. \end{cases}$$

Thus the equations for any rigid motion in the plane can be written as

$$\begin{cases} x = x' \cos\theta - y' \sin\theta + x_1 \\ y = x' \sin\theta + y' \cos\theta + y_1 \;. \end{cases}$$

If homogeneous coordinates are introduced, the system of equations takes the form of the linear transformation

$$\begin{cases} x = x' \cos\theta - y' \sin\theta + w'x_1 \\ y = x' \sin\theta + y' \cos\theta + w'y_1 \\ w = \qquad\qquad\qquad\qquad\quad w' \;. \end{cases}$$

The matrix which represents this transformation is

$$\begin{bmatrix} \cos\theta & \sin\theta & x_1 \\ \sin\theta & \cos\theta & y_1 \\ 0 & 0 & 1 \end{bmatrix} .$$

The determinant of this matrix is 1. The matrix obtained by deleting the last row and column of the above matrix is

$$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This is the matrix of the rotation of the coordinate axis about the origin in the plane. Since the vectors are mutually orthogonal and the norm of each vector is 1, this matrix is orthogonal.

IDENTIFICATION OF SECOND DEGREE CURVES

The following two theorems are needed for the identification of any second degree curve.

Theorem 2. If B is any matrix and A is a nonsingular matrix, then the rank of AB and BA are both equal to the rank of B (1).

Although the proof will not be given, the following results are implied by the theorem. If A and B are square matrices of the same order and if A^{-1} exists, then BA has the same rank and order as B. Since A is nonsingular implies A' is nonsingular, A'BA has the same rank as B. Similarly, ABA', $A^{-1}BA$, and ABA^{-1} have the rank as AB, and thus the same rank as B.

Theorem 3. Similar matrices have the same characteristic polynomial and the same determinant.

Proof: If $B = P^{-1}AP$, then

$$\left| B - \lambda I \right| = \left| P^{-1}AP - \lambda I P^{-1}P \right| = \left| P^{-1} \right| \left| A - \lambda I \right| \left| P \right| .$$

Since $\left| P^{-1} \right| = \dfrac{1}{|P|}$, the characteristic polynomials are equal.

By setting $\lambda = 0$ in the equation $\left| A - \lambda I \right| = f(\lambda)$, it is seen that the constant term of the characteristic polynomial of A is the determinant of A. But this is also the determinant of B so $\left| A \right| = \left| B \right|$.

Consider the general second degree equation in two variables with real coefficients. It can be written in the form

$$ax^2 + 2hxy + by^2 + 2px + 2qy + d = 0 . \qquad (6)$$

Upon introducing homogeneous coordinates, by replacing x by x/w and y by y/w, the equation becomes

$$a\frac{x^2}{w^2} + 2h\frac{x}{w}\frac{y}{w} + b\frac{y^2}{w} + 2p\frac{x}{w} + 2q\frac{y}{w} + d = 0$$

or

$$ax^2 + 2hxy + by^2 + 2pxw + 2qyw + dw^2 = 0 \qquad (7)$$

Equation (6) can be obtained from (7) by letting w = 1.

Equation (7) is a quadratic form which may be written as

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a & h & p \\ h & b & q \\ p & q & d \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0 \qquad (8)$$

To change the system of coordinates in (8) so as to eliminate the xy term, first consider the system of equations (5) written in matrix form. This becomes

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x_1 \\ \sin\theta & \cos\theta & y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \qquad (9)$$

Let

$$V = \begin{bmatrix} x \\ y \\ w \end{bmatrix}, \quad V_1 = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}, \quad S = \begin{bmatrix} \cos\theta & -\sin\theta & x_1 \\ \sin\theta & \cos\theta & y_1 \\ 0 & 0 & 1 \end{bmatrix} .$$

From (9)

$$V = SV_1 \quad \text{and} \quad V' = (SV_1)' = V_1'S' .$$

If the coefficient matrix of (8) is denoted by A, equation (8) can be written as $V'AV = 0$. The transformed equation is $V_1'S'ASV_1 = 0$, which, letting $B = S'AS$, becomes $V_1'BV_1 = 0$.

Let Q and T denote the two-by-two submatrices formed by deleting the last row and last column of A and S, respectively, so

$$Q = \begin{bmatrix} a & h \\ h & b \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where T is an orthogonal matrix. Let R be the two-by-two matrix formed by deleting the last row and last column of B. It is easily verified that R is independent of x_1 and y_1 and that $R = T'QT$. Since T is orthogonal, $T' = T^{-1}$. This implies $R = T^{-1}QT$, or that R is similar to Q.

From Theorems 1 and 2 it is seen that R and Q have the same rank, determinant, and characteristic equation. Also the rank and determinant of B are the same as those of A, since the determinant of S is equal to one.

Since Q is symmetric, T may be chosen so that R is in diagonal form. Let R be denoted by

$$\begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}$$

and B by

$$\begin{bmatrix} a' & 0 & p' \\ 0 & b' & q' \\ p' & q' & d' \end{bmatrix} .$$

Now $V_1'BV_1 = 0$ is

$$\begin{bmatrix} x' & y' & w' \end{bmatrix} \begin{bmatrix} a' & 0 & p' \\ 0 & b' & q' \\ p' & q' & d' \end{bmatrix} \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = 0 ,$$

or

$$a'x'^2 + b'y'^2 + 2p'x'w' + 2q'y'w' + d'w'^2 = 0 ,$$

whence replacing w' by 1, (6) becomes

$$a'x'^2 + b'y'^2 + 2p'x' + 2q'y' + d' = 0 . \tag{10}$$

The transformation has eliminated the xy term.

Consider the following standard forms of the equations of second degree in two variables:

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the real ellipse,

(ii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1$, the imaginary ellipse,

(iii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0$, imaginary intersecting straight lines,

(iv) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1,$ the hyperbola,

(v) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0,$ real intersecting straight lines,

(vi) $x^2 + 2py = 0,$ the parabola,

(vii) $x^2 + a^2 = 0,$ imaginary parallel lines,

(viii) $x^2 - a^2 = 0,$ real parallel lines,

(ix) $x^2 = 0,$ two coincident lines.

If these are to be considered as quadratic forms, the equations must be put into the form of equation (10). Equations (vi)-(ix) are already in this form. To change equations (i)-(v), all that is necessary is to multiply both sides of the equation by a^2b^2, and collect the nonzero terms on one side.

The following table exhibits the matrix B of the quadratic form and its rank for each of the nine cases.

Second degree equation	B	Rank
(i) The real ellipse	$\begin{bmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & -a^2b^2 \end{bmatrix}$	3
(ii) The imaginary ellipse	$\begin{bmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2b^2 \end{bmatrix}$	3
(iii) Imaginary intersecting straight lines	$\begin{bmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	2

(iv)	The hyperbola	$\begin{bmatrix} b^2 & 0 & 0 \\ 0 & -a^2 & 0 \\ 0 & 0 & -a^2b^2 \end{bmatrix}$	3
(v)	Real intersecting straight lines	$\begin{bmatrix} b^2 & 0 & 0 \\ 0 & -a^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	2
(vi)	The parabola	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & p \\ 0 & p & 0 \end{bmatrix}$	3
(vii)	Imaginary parallel lines	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$	2
(viii)	Real parallel lines	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a^2 \end{bmatrix}$	2
(ix)	Two coincident lines	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	1

Since the matrix B is of rank three for the ellipse, hyperbola, and parabola, A is also of rank three for these cases. Likewise, A is of rank two for distinct straight lines and rank one for coincident lines.

Further information valuable for the identification of the curve given by a second degree equation can be obtained from

the characteristic roots of the matrix R. For cases (i)-(iii) the matrix $R = \begin{bmatrix} b^2 & 0 \\ 0 & a^2 \end{bmatrix}$ and $\left| R \right| = b^2a^2$. For (iv) and (v) $R = \begin{bmatrix} b^2 & 0 \\ 0 & -a^2 \end{bmatrix}$ and $\left| R \right| = -b^2a^2$, and for (vi)-(ix) $R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\left| R \right| = 0$.

Since R is a diagonal matrix, $\left| R \right|$ is the product of the diagonal elements of R. The diagonal elements, however, are also the characteristic roots of R. Thus recalling $\left| R \right| = \left| Q \right|$, it is seen that the product of the characteristic roots of Q is positive for (i)-(iii), negative for (iv) and (v), and zero for (vi)-(ix).

In addition, if λ_1 and λ_2 are the characteristic roots of R, then $\left| R \right| = \left| Q \right| = \lambda_1\lambda_2 = ab - h^2$. Thus, provided the matrix B is of rank three, an irreducible second degree curve is an ellipse if $ab - h^2 > 0$, a hyperbola is $ab - h^2 < 0$, and a parabola if $ab - h^2 = 0$.

Additional information for identification of a second degree curve can be obtained by consideration of the matrix $B = S'AS$

$$B = S'AS = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ x_1 & y_1 & 1 \end{bmatrix} \begin{bmatrix} q & h & p \\ h & b & q \\ p & q & d \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & x_1 \\ \sin\theta & \cos\theta & y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Let this product be denoted by a three-by-three matrix

$$\begin{bmatrix} a' & h' & p' \\ h' & b' & q' \\ p' & q' & d' \end{bmatrix} ,$$

where

$$a' = a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta$$
$$h' = (b - a)\sin \theta \cos \theta + h(\cos^2 \theta - \sin^2 \theta)$$
$$b' = a \sin^2 \theta - 2h \sin \theta \cos \theta + b \cos^2 \theta .$$

Since h' must be zero for the xy term to vanish, θ is determined. Further, for such a value of θ,

$$p' = (a \cos \theta + h \sin \theta)x_1 + (h \cos \theta + b \sin \theta)y_1$$
$$+ p \cos \theta + q \sin \theta$$
$$q' = (-a \sin \theta + h \cos \theta)x_1 + (-h \sin \theta + b \cos \theta)y_1$$
$$- p \sin \theta + q \cos \theta$$
$$d' = ax_1{}^2 + 2hx_1y_1 + by_1{}^2 + 2px_1 + 2qy_1 + d_1 .$$

It will be possible to choose x_1 and y_1 so that p' and q' both become zero if the determinant of the coefficients of x_1 and y_1 in these two equations does not vanish. These choices give

$$\begin{vmatrix} a \cos \theta + h \sin \theta & h \cos \theta + b \sin \theta \\ -a \sin \theta + h \cos \theta & -h \sin \theta + b \cos \theta \end{vmatrix} = ab - h^2 \neq 0.$$

Since the determinant of Q is $ab - h^2$, the above result implies that the rank of Q is two. If x_1, y_1 and θ are chosen such that $h' = p' = q' = 0$, the matrix B is of the form

$$\begin{bmatrix} a' & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & d' \end{bmatrix}$$

Hence

$$V_1{}'BV_1 = \begin{bmatrix} x' & y' & w' \end{bmatrix} \begin{bmatrix} a' & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & d' \end{bmatrix} \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$

or

$$a'x'^2 + b'y'^2 + d'w'^2 = 0 .$$

Now letting $w' = 1$, the equation becomes

$$a'x'^2 + b'y'^2 + d' = 0.$$

Dividing through by d', this becomes

$$\frac{a'}{d'} x'^2 + \frac{b'}{d'} y'^2 + 1 = 0$$

which, depending on the value of d', is one of the forms (i) to (v).

If $ab - h^2 = 0$, it will be shown that $a + b \neq 0$, and that the angle θ which makes $h' = 0$ can be chosen such that

$\cos\theta = \sqrt{\dfrac{a}{a + b}}$ and $\sin\theta = \pm\sqrt{\dfrac{b}{a + b}}$, where the sign is the same as that of ah.

To prove that $a + b \neq 0$ if $ab - h^2 = 0$, assume that $ab - h^2 = 0$ and $a + b = 0$. Then $a = -b$, and $-b^2 - h^2 = 0$, or $h = \pm\, i\sqrt{b}$. This contradicts the assumption that (6) had real coefficients. Therefore $a + b \neq 0$.

If $\cos\theta = \sqrt{\dfrac{a}{a + b}}$ and ah is positive, $\sin\theta = \sqrt{\dfrac{b}{a + b}}$,

$$h' = (b - a)\sqrt{\frac{b}{a + b}}\sqrt{\frac{a}{a + b}} + h(\frac{a}{a + b} - \frac{b}{a + b})$$

$$= \frac{b\sqrt{ab} - a\sqrt{ab} + ha - hb}{a + b}$$

$$= \frac{bh - ah + ah - bh}{a + b}$$

$$= 0$$

Similarly, it can be shown that if ah is negative,

$\sin \theta = -\sqrt{\frac{b}{a + b}}$ and $\cos \theta = \sqrt{\frac{a}{a + b}}$ will make $h' = 0$.

In addition it can be shown that (1) these restrictions will give

$$a' = a + b, \quad b' = 0$$

$$p' = x_1\sqrt{a(a + b)} \pm y_1\sqrt{b(a + b)} + p\sqrt{\frac{a}{a + b}} \pm q\sqrt{\frac{b}{a + b}}$$

and

$$q' = q\sqrt{\frac{a}{a + b}} - (\pm p\sqrt{\frac{b}{a + b}}) .$$

There are two cases to consider, $q' \neq 0$, and $q' = 0$.

If $q' \neq 0$, then $d' = (\frac{1}{a + b})p'^2 + \ell x_1 + my_1 + n$, and it can be shown that x_1 and y_1 may be chosen so p' and d' are zero. This would reduce equation (10) to the form (vi).

The last case to consider is that for which $q' = 0$. If

$q' = 0$, then $d' = \dfrac{1}{a+b} p'^2 + r$, and x_1 and y_1 may be chosen so that $p' = 0$. Such a choice will give a value of d' which can be shown to be independent of the particular choice of x_1 and y_1 and equation (10) will reduce to one of the forms (vii), (viii), or (ix). The matrix R and thus the matrix Q, is of rank two for forms (i)-(v), and of rank one for forms (vi)-(ix).

With the preceding results, complete identification of any second degree curve in two variables can be made by considering the matrices A and Q. The identification depends upon the ranks of the matrices A and Q, and on the characteristic equation of Q. In any case, a matrix S can be found which will reduce equation (6) to one of the standard forms.

As an example of these procedures, again consider the conic $2x^2 + 4xy + 5y^2 = 1$. Introducing homogeneous coordinates, replacing x by x/w and y by y/w, the equation becomes $2x^2 + 4xy + 5y^2 - w^2 = 0$, or in matrix form

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

The matrices A and Q are

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix},$$

which are clearly of rank three and two, respectively. From the

ranks of A and Q it is seen the conic must be an ellipse or a hyperbola. But since the characteristic roots of Q are 6 and 1 and the product of the characteristic roots is greater than zero, the conic must be an ellipse. Alternatively, one could compute $ab - h^2$, which in this case is $(2)(5) - 4 = 6$. Since this is greater than zero and the rank of A is three, the conic must be an ellipse.

SECOND DEGREE SURFACES

The general second degree equation in three variables x, y, and z can be written

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy + 2px + 2qy + 2rz + d = 0 \tag{11}$$

where not all of a, b, c, f, g, h are zero. Again homogeneous coordinates are introduced, replacing x by x/w, y by y/w, and z by z/w. The homogeneous equation is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy + 2pxw + 2qyw + 2rzw + dw^2 = 0\ . \tag{12}$$

The derivation of properties useful for identifying the particular surface parallels that of the previous section. Equation (12) may be written as

$$\begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

where the coefficient matrix is denoted by A. In this case

$$Q = \begin{bmatrix} a & h & q \\ h & b & f \\ g & f & c \end{bmatrix} .$$

The matrix S needed to find B = S'AS will be of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & x_1 \\ a_{21} & a_{22} & a_{23} & y_1 \\ a_{31} & a_{32} & a_{33} & z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} ,$$

is orthogonal.

Then if the transformed equation of (11) is

$$a'x'^2 + b'y'^2 + c'z'^2 + 2f'y'z' + 2g'x'z' + 2b'x'y'$$
$$+ 2p'x' + 2q'y' + 2r'z' + d' = 0 ,$$

the matrix R is of the form

$$\begin{bmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{bmatrix} .$$

As before the rank and characteristic equation of R are

the same as those of Q, while the rank and determinant of B are the same as those of A.

As illustrations consider the following three equations selected from the set of standard equations of the second degree.

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1 = 0,$ the real ellipsoid,

(ii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + 2rz = 0,$ the elliptic paraboloid,

(iii) $x^2 - a^2 = 0,$ real parallel planes.

Following is a table of the matrices B and R and their ranks for the three cases.

	B	Rank	R	Rank
(i)	$\begin{bmatrix} b^2c^2 & 0 & 0 & 0 \\ 0 & a^2c^2 & 0 & 0 \\ 0 & 0 & a^2b^2 & 0 \\ 0 & 0 & 0 & -a^2b^2c^2 \end{bmatrix}$	4	$\begin{bmatrix} b^2c^2 & 0 & 0 \\ 0 & a^2c^2 & 0 \\ 0 & 0 & a^2b^2 \end{bmatrix}$	3
(ii)	$\begin{bmatrix} b^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & 0 & a^2b^2r \\ 0 & 0 & a^2b^2r & 0 \end{bmatrix}$	4	$\begin{bmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	2
(iii)	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^2 \end{bmatrix}$	2	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	1

Note that the product of the characteristic roots of R is greater than or equal to zero for all three cases.

It can be shown that every equation of the form (11) can be reduced to a standard form of a surface, while various properties of the matrices A and Q remain unchanged. Thus determination of the ranks of A and Q and the characteristic equation and characteristic roots of Q will identify the equation (11).

Theoretically, a similar process could be carried out in Euclidean spaces of any dimension. However, as the dimension of the space increases, the order of the matrices involved will render this method impractical in view of the computations involved. Major difficulties arise first in finding the unitary or orthogonal matrix which will change the quadratic form to canonical form, and then in finding the characteristic equation and characteristic roots of the matrices denoted above as A and Q. These can be found by indirect methods, but such methods usually involve approximation techniques which although they are convenient in that they are applicable to computers, often lead to inaccuracies in the results acquired. Caution must be taken, therefore, to assure that one does not misinterpret the results thus obtained.

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QUADRATIC FORMS AND APPLICATIONS TO GEOMETRY

by

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The purpose of this report is to consider applications of quadratic forms to geometry. The report begins with a section containing definitions essential to the development of the discussion. Then utilizing the definitions of characteristic and similar matrices, a method for changing a matrix to diagonal form is developed.

Next, using the concepts of orthogonality and similarity, a procedure is derived for reducing a quadratic form to canonical form. An example is given to illustrate this procedure.

Before consideration of the change of a coordinate system, homogeneous coordinates are introduced as a means of simplifying the procedure. With the homogeneous coordinates, the coordinate transformation then takes the form of a linear transformation.

The next step is to utilize a combination of the preceding concepts and procedures to derive various properties useful for identifying any second degree curve. From the ranks and determinants of the matrices involved, a complete identification of the conic from the equation is possible.

Finally, the extension of these methods to second degree surfaces is indicated. In this section examples of the general quadratic form and the matrices involved for three types of second degree surfaces are presented.